

Analytical Expression of Time Domain Response of Arbitrary Order Fractance

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Abstract— Fractance is the terminology for any integro-differential operator which its order can be fractional. It is often cited in various disciplines such as biomedical engineering, control engineering and electronic engineering etc., which are the crucial basis in the design and implementation of signal processing systems. Since analytically obtaining time domain response of fractance yields advantage in many circumstances, an analytical expression of such response has been derived in this research where the fractance under consideration can be of arbitrary order even though the magnitude of such order may be greater than unity. This expression has been found to be an efficient mathematical tool for those aforementioned fractance involved disciplines which support the signal processing system design and implementation.

I. INTRODUCTION

Fractance i.e. fractional impedance is the analog domain terminology for any integro-differential operator which its order, α can be fractional. Fractance is often cited in various disciplines for example biomedical engineering [1]-[3], control engineering [4]-[6] and electronic engineering [7]-[9] etc., which are the crucial basis in the design and implementation of signal processing systems. In many circumstances, analytically obtaining the time domain responses of such fractance yields an advantage. An example of such circumstances is the characterizations of biological tissues [1]-[3], [10], [11] which can be considered in the analog domain as fractances. Traditionally, such characterizations can be performed by using the numerical simulation [12] based on the measured time domain responses obtained from exciting the target fractance by the predetermined inputs. This traditional methodology is cumbersome and also yields non analytical results which are imprecise compared to the analytical ones. It can be seen that much of the effort can be reduced and the analytical characterization results can be expected if the time domain responses of the fractance under consideration can be analytically obtained. According to various citations of fractance and such advantage of analytically obtaining its time domain responses, the analytical expressions of these responses have been proposed in previous studies such as [13], [14] and [15] etc. Unfortunately, only fractance with $|\alpha| \leq 1$ has been focused in these studies. For example, only fractance with $\alpha = 0.5$ and super capacitor i.e. fractional capacitor which $|\alpha| \leq 1$,

have been respectively considered in [13] and [14]. In [15], the fractional trigonometric functions based expression of sinusoidal response of fractance has been derived under the assumption that $|\alpha| \leq 1$.

Hence, an analytical expression of voltage response in time domain of fractance has been derived in this research under the assumption that the often cited step function has been chosen as the current input according to its simplicity and mathematical relationships to other renowned inputs such as impulse function, ramp function and parabolic function etc. The resulting expression can be applied to fractance with arbitrary α even though $|\alpha| > 1$. By the aforementioned mathematical relationships, expressions of responses to other renowned inputs mentioned above can be derived by using the proposed expression as the basis. So, the derived expression has been found to be an efficient mathematical tool for those fractance interested research fields which are beneficial to the design and implementation of signal processing systems.

II. DERIVATION OF THE EXPRESSION

In this section, derivation of the proposed expression will be shown. For any fractance, its impedance function, $Z(s)$ can be given by

$$Z(s) = Ks^\alpha \quad (1)$$

where K denote the amplitude of $Z(s)$. By using (1), the s-domain voltage response, $V(s)$ can be found as

$$V(s) = Ks^\alpha I(s) \quad (2)$$

where $I(s)$ denotes the s-domain current input.

By taking the inverse Laplace transformation, the desired time domain voltage response, $V(t)$ can be obtained as follows

$$V(t) = KD^\alpha [I(t)] \quad (3)$$

where $D^\alpha []$ denotes the fractional order derivative operator. There exist various mathematical definitions of fractional order derivative [16]-[18] etc. For this research, the renowned

Riemann-Liouville definition has been generically used i.e. without choosing any specific value of m which defines the lower bound and upper bound of α . So, m can be any real integer.

By using such generic definition of fractional order derivative, $V(t)$ become

$$V(t) = \frac{K}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} I(\tau) d\tau \quad (4)$$

where $m-1 < \alpha \leq m$. Under the aforementioned assumption, $I(t) = I_0 u(t)$ where I_0 denotes the amplitude of $I(t)$. As a result, $V(t)$ after the integration is

$$V(t) = \frac{KI_0}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left[\frac{t^{m-\alpha}}{m-\alpha} u(t) \right] \quad (5)$$

where $m-1 < \alpha \leq m$.

By evaluating the derivative in (5) in the iterative manner and baring the fact that m can be negative which the derivative become integral in mind, $V(t)$ can be finally derived as follows

$$V(t) = \begin{cases} \frac{-KI_0}{\alpha\Gamma(m-\alpha+1)} \left[\prod_{n=0}^m (n-\alpha) \right] t^{-\alpha} u(t) & ; m \in \{0\} \cup I^+ \\ \frac{KI_0}{\Gamma(m-\alpha)} \left[\prod_{n=0}^{|m|} (n+m-\alpha) \right]^{-1} t^{-\alpha} u(t) & ; m \in \{I^-\} \end{cases} \quad (6)$$

where $m-1 < \alpha \leq m$.

By using the derived expression, $V(t)$ of fractance with any α can be analytically obtained even though such fractance may has $|\alpha| > 1$. This is because m can be any real integer so, there are infinitely possible ranges for α as $m-1 < \alpha \leq m$. As a result, α can take any real value. This versatility of the derived expression is caused by the usage of the Riemann-Liouville definition in the generic manner as mentioned above.

As illustrations, determining of $V(t)$ of both fractance with $|\alpha| \leq 1$ and with $|\alpha| > 1$ will be demonstrated. Firstly, that with $|\alpha| \leq 1$ such as that with $-1 < \alpha \leq 0$ i.e. $m = 0$, for example, the celebrated half order integrator/capacitor which $\alpha = -0.5$ etc., will be considered. Here, $V(t)$ of fractance with $-1 < \alpha \leq 0$ can be found by using the proposed expression with $m = 0$ as

$$V(t) = \frac{KI_0 t^{-\alpha} u(t)}{\Gamma(1-\alpha)} \quad (7)$$

One may argue that the proposed expression is eventually reduced to (7) which is much simpler. Unfortunately, this reduction is not possible. For this case which $m \in \{0\} \cup I^+$, some necessary conditions such as $m-\alpha-1 > 0$, $m-\alpha-2 > 0$ and $m-\alpha-3 > 0$ etc., violate $m-1 < \alpha \leq m$ since they are $\alpha < m-1$,

$\alpha < m-2$ and $\alpha < m-3$ etc., respectively. As a result, the proposed expression is not reducible.

Now, fractance with $|\alpha| > 1$ such as that with $-2 < \alpha \leq -1$ and $-3 < \alpha \leq -2$ i.e. $m = -1$ and $m = -2$ respectively, will be considered. For fractance with $-2 < \alpha \leq -1$, $V(t)$ can be found by using the proposed expression with $m = -1$ as shown in (8). When $K = 1$ and $I_0 = 1$, this $V(t)$ can be plotted against t and α as in Fig.1. One may also argue that (8) can be reduced to (7) if and only if necessary constraints including $\alpha > -1$ are met. Unfortunately, $\alpha > -1$ cannot be met since the range of α for this circumstance is $-2 < \alpha \leq -1$. So, (8) is not reducible.

$$V(t) = \frac{KI_0 t^{-\alpha} u(t)}{\alpha(1+\alpha)\Gamma(-1-\alpha)} \quad (8)$$

For fractance with $-3 < \alpha \leq -2$, $V(t)$ can be found by using the proposed expression with $m = -2$ as shown in (9). When $K = 1$ and $I_0 = 1$, this $V(t)$ can be plotted against t and α as depicted in Fig.2. Once again, one may argue that (9) can be reduced to (7). However, this reduction is impossible since $\alpha > -2$ which is one of the necessary conditions, cannot be met as $-3 < \alpha \leq -2$ in this circumstance.

$$V(t) = \frac{KI_0 t^{-\alpha} u(t)}{\alpha(\alpha+1)(\alpha+2)\Gamma(-2-\alpha)} \quad (9)$$

At this point, it can be seen that there are many possible $V(t)$ up to the chosen ranges of α such as (7), (8) and (9) etc., which can be conveniently formulated by using the proposed expression. So, this expression has been found to be an efficient and necessary mathematical tool for various fractance involved disciplines mentioned above which are the very important foundations in the design and implementation of signal processing systems. In the subsequent section, practical applications of this expression to specific fractances will be presented

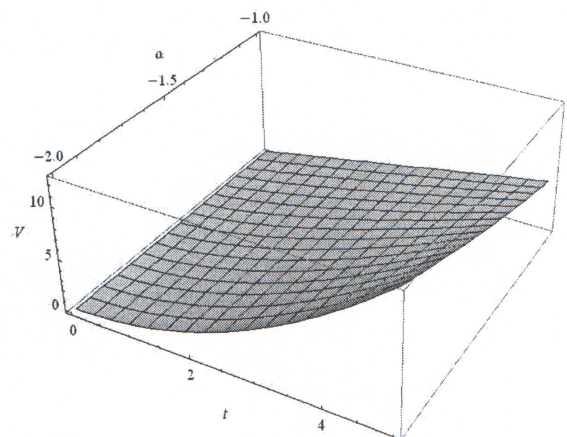


Fig. 1. $V(t)$ of fractance with $-2 < \alpha \leq -1$ v.s. t and α when $K = 1$ and $I_0 = 1$

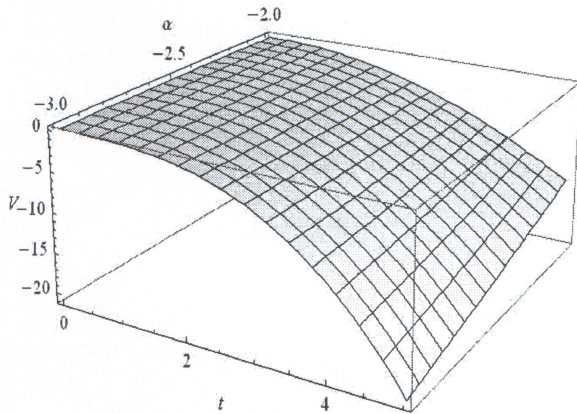


Fig. 2. $V(t)$ of fractance with $-3 < \alpha \leq -2$ v.s. t and α when $K = 1$ and $I_0 = 1$

III. PRACTICAL APPLICATION

Firstly, the half order integrator/capacitor will be considered. Since its $\alpha = -0.5$ i.e. $m = 0$ and $|\alpha| \leq 1$, its $V(t)$ can be determined by using the proposed expression with such m and α as shown in (10) and can be plotted with respect to t as depicted in Fig.3 when $K = 1$ and $I_0 = 1$.

$$V(t) = 2KI_0 \sqrt{\frac{t}{\pi}} u(t) \quad (10)$$

Now, the fractance with $\alpha = 0.5$ which is also known as the renowned half order differentiator/inductor will be considered. Since $\alpha = 0.5$ i.e. $m = 1$ and $|\alpha| \leq 1$, $V(t)$ of the half order differentiator/inductor can be found by using the proposed expression with these m and α as shown in (11) where its similarity to the step response proposed in [13] can be observed.

$$V(t) = \frac{KI_0 u(t)}{\sqrt{\pi t}} \quad (11)$$

Finally, the fractance with $\alpha = -1.5$ i.e. $m = -1$ and $|\alpha| > 1$, will be considered. By using the proposed expression with these m and α , $V(t)$ of this fractance can be found as shown in (12) and can be plotted with respect to t when $K = 1$ and $I_0 = 1$ as depicted in Fig.4. If α become -2 , the Frequency Dependent Negative Resistor (FDNR) is obtained and $V(t)$ can be given by $V(t) = 0.5KI_0 t^2 u(t)$.

$$V(t) = \frac{4KI_0}{3} \sqrt{\frac{t^3}{\pi}} u(t) \quad (12)$$

In the next section, the proposed expression based derivation of the voltage responses to other renowned inputs i.e. impulse function, ramp function and parabolic function will be demonstrated.

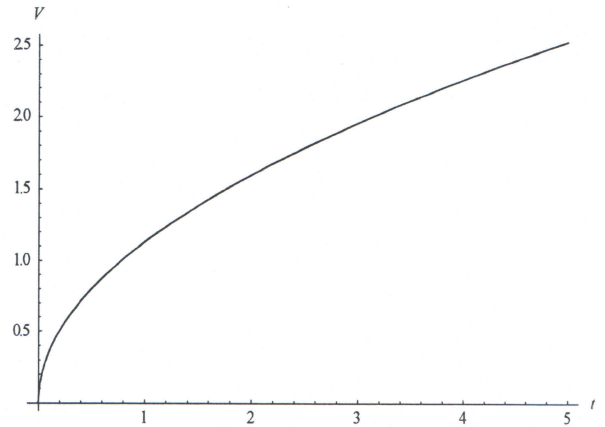


Fig. 3. $V(t)$ of half order integrator/capacitor v.s. t when $K = 1$ and $I_0 = 1$

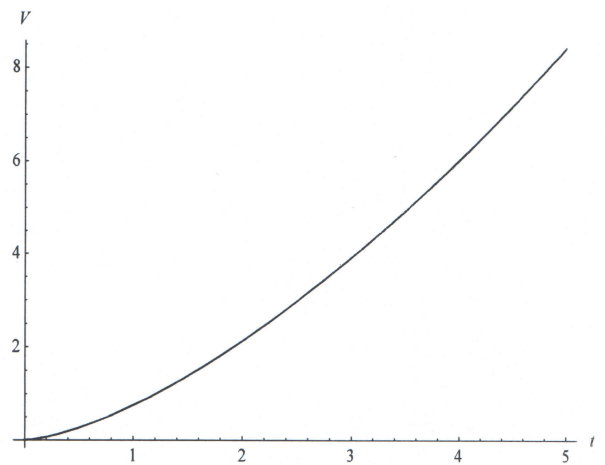


Fig. 4. $V(t)$ of fractance with $\alpha = -1.5$ v.s. t when $K = 1$ and $I_0 = 1$

IV. VOLTAGE RESPONSES TO OTHER RENOWNED INPUTS

Since the impulse function input given by $I_\delta(t) = I_0 \delta(t)$ can be related to $I(t) = I_0 u(t)$ by $I_\delta(t) = D[I(t)]$ where $D[]$ denotes the 1st order differential operator, the voltage response of fractance of order α to $I_\delta(t)$, $V_\delta(t)$ can be given based on the proposed expression as

$$V_\delta(t) = D[V(t)] = \begin{cases} \frac{KI_0 \left[\prod_{n=0}^m (n - \alpha) \right] t^{-\alpha-1} u(t)}{\Gamma(m - \alpha + 1)} & ; m \in \{0\} \cup I^+ \\ -\alpha KI_0 \left[\prod_{n=0}^{|m|} (n + m - \alpha) \right]^{-1} t^{-\alpha-1} u(t) & ; m \in \{I^-\} \end{cases} \quad (13)$$

where $m-1 < \alpha \leq m$. A similar voltage response can be obtained by applying $I(t)$ to the fractance of order $\alpha + 1$ which can be viewed as that of order α cascaded by an ordinary

differentiator/inductor. Obviously, one may claim that $V_\delta(t)$ as shown in (13) can be eventually reduced to

$$V_\delta(t) = \frac{KI_0 t^{-\alpha-1} u(t)}{\Gamma(-\alpha)} \quad (14)$$

which is much simpler. Similarly to the proposed expression, $V_\delta(t)$ as in (13) is not reducible because some necessary conditions such as $m-\alpha-1 > 0$, $m-\alpha-2 > 0$ and $m-\alpha-3 > 0$ etc., clearly violate $m-1 < \alpha \leq m$ as they are $\alpha < m-1$, $\alpha < m-2$ and $\alpha < m-3$ etc., respectively.

As the ramp function input given by $I_r(t) = I_0 t u(t)$ can be related to $I(t)$ by $I_r(t) = D^{-1}[I(t)]$ where $D^{-1}[\]$ denotes the 1st order integral operator, the voltage response of fractance of order α to $I_r(t)$, $V_r(t)$ can be given based on the proposed expression as

$$V_r(t) = D^{-1}[V(t)] = \begin{cases} \frac{KI_0 \left[\prod_{n=0}^m (n-\alpha) \right] t^{-\alpha+1} u(t)}{\alpha(\alpha-1)\Gamma(m-\alpha+1)} & ; m \in \{0\} \cup I^+ \\ \frac{KI_0 \left[\prod_{n=0}^{|m|} (n+m-\alpha) \right]^{-1} t^{-\alpha+1} u(t)}{(1-\alpha)\Gamma(m-\alpha)} & ; m \in \{I^-\} \end{cases} \quad (15)$$

where $m-1 < \alpha \leq m$. A similar response can be obtained by applying $I(t)$ to the fractance of order $\alpha - 1$ which can be viewed as that of order α cascaded with an ordinary integrator/capacitor.

Finally, since the parabolic function input given by $I_p(t) = I_0 t^2 u(t)$ can be related to $I(t)$ by $I_p(t) = D^{-2}[I(t)]$ where $D^{-2}[\]$ denotes the 2nd order integral operator, the voltage response of fractance of order α to $I_p(t)$, $V_p(t)$ can be given based on the proposed expression as follows

$$V_p(t) = D^{-2}[V(t)] = \begin{cases} \frac{-KI_0 \left[\prod_{n=0}^m (n-\alpha) \right] t^{-\alpha+2} u(t)}{\alpha(\alpha-1)(\alpha-2)\Gamma(m-\alpha+1)} & ; m \in \{0\} \cup I^+ \\ \frac{KI_0 \left[\prod_{n=0}^{|m|} (n+m-\alpha) \right]^{-1} t^{-\alpha+2} u(t)}{(\alpha-1)(\alpha-2)\Gamma(m-\alpha)} & ; m \in \{I^-\} \end{cases} \quad (16)$$

where $m-1 < \alpha \leq m$. A similar response can be obtained from by applying $I(t)$ to fractance of order $\alpha - 2$ which can be viewed as that with order α cascaded by two normal integrators/capacitors.

V. CONCLUSION

In this research, an analytical expression in time domain of voltage response to the often cited step input current of fractance has been derived as analytically obtaining time domain response of fractance yields advantage in many

circumstances. This expression can be applied to fractance of arbitrary α even though $|\alpha| > 1$. The responses to other renowned inputs such as impulse, ramp and parabola can be analytically determined based on the proposed expression as demonstrated. So, this expression has been found to be an efficient mathematical tool for the aforementioned fractance involved disciplines which support the design and implementation of signal processing systems.

ACKNOWLEDGEMENT

The first author would like to acknowledge Mahidol University, Thailand for online database service.

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